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2001 J. Phys. A: Math. Gen. 34 2007

(http://iopscience.iop.org/0305-4470/34/10/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2007–2022 www.iop.org/Journals/ja PII: S0305-4470(01)20422-3

Ghost components in the Jost function and a new class of phase-equivalent potentials

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Received 16 June 2000, in final form 19 December 2000

Abstract
We show that the introduction of components, in the Jost function, that create new bound states while leaving the S-matrix unchanged, generates potentials behaving as r^{-2} at large distances. We demonstrate that the modified Jost functions can be obtained by applying two successive supersymmetric transformations to the original potential. We further show that transparent potentials, with $S_\ell(k) \equiv 1$, can also be obtained by successive supersymmetric transformations. They are characterized by the property that their SUSY-2 partners resemble centrifugal barriers. Finally, the relation of these transformations to the asymptotic normalization constants of the inverse scattering problem is discussed. We show that the two supersymmetric transformations that remove a bound state provide a potential which is the same as that obtained via the Marchenko inverse scattering procedure, when the asymptotic normalization constant is set to zero.

PACS number: 1130

1. Introduction

The quantum mechanical inverse scattering problem was formulated a long time ago (see, for example, [1, 2] and references therein). For the class of central potentials $V(r)$ satisfying the integrability conditions

$$\begin{aligned} &\int_0^\infty r|V(r)|\,\mathrm{d}r < \infty \\ &\int_b^\infty V(r)\,\mathrm{d}r < \infty \qquad b > 0 \end{aligned} \tag{1}$$

it is well established that the knowledge of the corresponding scattering phase shifts, i.e. of the S-function $S_\ell(k)$, at all energies for a fixed angular momentum ℓ, allows the reconstruction of the underlying potential which, in the absence of bound states, is unique [3]. When the potential supports bound states, its determination also requires knowledge of the bound state energies and of the corresponding normalization constants.

In this work we shall address a peculiar aspect of the inverse scattering problem, namely, the possible existence of ghost components in the Jost function, i.e. components that leave the scattering matrix unchanged, and their consequences on the scattering potential. We recall that the S-matrix is defined by the ratio

$$S_\ell(k) = \frac{F_\ell(-k)}{F_\ell(k)} \tag{2}$$

where $F_\ell(k)$ is the Jost function which contains the physical information. Thus, for example, the zeros of the Jost function in the upper complex k-axis correspond to bound states, while its knowledge at other points of the k-plane can provide us the phase shifts (real k-axis), the resonances (zeros in the fourth quadrant), etc. In contrast, the above definition of the S-matrix, can introduce spurious or unphysical features. A pole in the S-matrix, for instance, could be attributed to a zero of the Jost function $F_\ell(k)$ or to a pole of $F_\ell(-k)$. In the former case, the pole in the S-matrix is a genuine physical bound state while in the latter, it is spurious. Moreover, equation (2) suggests that ghost components in the Jost function could be introduced by multiplying the Jost function with a real even function of k with the constraints that (a) the Jost function must asymptotically go to unity as $k \to \infty$ and (b) it is analytic in the upper half k-plane in order to preserve the analyticity properties of the Jost function. It is interesting to investigate the new class of phase-equivalent potentials generated by such a modification.

The present paper deals with this new class of potentials. In section 2 we give an explicit expression of the Bargmann ghost components and we calculate the corresponding transparent potentials and regular wavefunctions. In section 3 we present some analytical examples for transparent potentials. In section 4 we generalize the derivation of phase-equivalent potentials to a non-zero potential. Our conclusions are drawn in section 5. Some mathematical details are shifted to appendices A and B.

2. Construction of transparent potentials

As mentioned above, the ghost components, henceforth denoted by $F_\ell^G(k)$, must asymptotically go to unity. Assuming that they may be represented by rational functions

$$F_\ell^G(k) = \prod_{n=1}^{N_G} \frac{k + \mathrm{i}\gamma_n}{k + \mathrm{i}b_n} \qquad \gamma_n, b_n \text{ real} \tag{3}$$

the N_G parameters b_ns must satisfy $b_n \geqslant 0$ in order to preserve the analyticity of the Jost function in the upper half k-plane. By definition F_ℓ^G is real and thus

$$F_\ell^G(k) = \prod_{n=1}^{N_G} \frac{k + \mathrm{i}\gamma_n}{k + \mathrm{i}b_n} = \prod_{n=1}^{N_G} \frac{k - \mathrm{i}\gamma_n}{k - \mathrm{i}b_n}. \tag{4}$$

From the analyticity of F_ℓ^G we have $b_n = 0$ for all n. On the other hand, $\mathrm{i}\gamma_n$ and $-\mathrm{i}\gamma_n$ are roots of $F_\ell^G(k) = 0$. Hence, N_G must be even, $N_G = 2N$, and the ghost component can be written as

$$F_\ell^G(k) = \prod_{n=1}^{N} \frac{k^2 + \gamma_n^2}{k^2}. \tag{5}$$

At this stage we would like to emphasize that the introduction of a ghost component, in the Jost function, introduces a bound state (a zero of the Jost function) which does not manifest itself in the S-matrix, which is directly linked to experiment. Since the phase shifts remained unchanged, Levinson's theorem, $\delta(0) - \delta(\infty) = N_B\pi$, with N_B being the number of bound states, is no longer applicable and one expects a new potential, not belonging to class I, characterized by the integrability conditions (1), and at least as singular as $1/r^2$ at short distances. For such potentials it was shown long ago by Swan [4] that one may have πs, in phase shifts, which are not associated with bound states.

Note that the constraint (b_n real) can be released. The case where the Jost function may have spectral singularities at the points $k = \pm b$ has been investigated by Kurasov [5] and will be briefly discussed at the end of section 4.

The potentials associated with the transformed Jost function can be extracted from the potentials in the Faddeev class induced by

$$F_\ell(k) = \prod_{i=1}^{N} \frac{k^2 + \gamma_n^2}{(k + \mathrm{i}\epsilon)^2}$$

within the framework of the Marchenko theory. According to Kurasov's [6] conjecture, the limit as ϵ goes to zero, would produce the desired potentials and regular solutions.

However, this procedure becomes rapidly intractable as the number N of components in (5) increases, and we have used the following theorem:

Theorem. *Let $V(r)$ be a potential which, in the ℓ-partial wave, supports N bound states, of energies $E_n = -\hbar^2\gamma_n^2/2m$ ($n = 1, \ldots, N$), and depends on N asymptotic normalization constants C_n, associated with the energies E_n, and defined by*

$$C_n^{-1} = \int_0^\infty \psi_\ell(\mathrm{i}\gamma_n, r)^2\, \mathrm{d}r \tag{6}$$

where ψ_ℓ denotes the regular solution of the Schrödinger equation at $k = \mathrm{i}\gamma_n$, which satisfies the boundary condition

$$\lim_{r\to 0} \psi_\ell(k, r)\,(2\ell + 1)!!/r^{\ell+1} = 1. \tag{7}$$

When one of the normalizations, say C_p, tends to infinity, the potential becomes singular at the origin behaving like $2(2\ell + 3)/r^2$ and the original Jost function is multiplied by the factor of $k^2/(k^2 + \gamma_p^2)$ which cancels out the ghost component.

A proof for this theorem is given in appendix A. The resulting Jost function

$$\tilde{F}_{\ell+2}(k) = \frac{k^2}{k^2 + \gamma_p^2} F_\ell(k) \tag{8}$$

has behaviour that corresponds to the $(\ell + 2)$th-partial wave. This transformation of the Jost function, equation (8), is equivalently the result of the product of the supersymmetric (SUSY) transformations T_1 and T_3 of Sukumar [7]. Transformation T_1 changes the Jost function by the multiplicative term $k/(k - \mathrm{i}\gamma_p)$ and increases ℓ to $\ell + 1$. Similarly, T_3 changes the Jost function by the multiplicative term $k/(k + \mathrm{i}\gamma_p)$ and changes ℓ to $\ell + 1$ (for details see [7]). We note that the construction of phase-equivalent supersymmetric potentials, with a $1/r^2$ singularity at the origin and which have the same behaviour at infinity as the original potential, was suggested some years ago by Baye [8].

When the above transformation, equation (8), is applied successively for all energies of the discrete spectrum, it leads to the transformed Jost function

$$\tilde{F}_{\ell+2N}(k) = \prod_{n=1}^{N} \frac{k^2}{k^2 + \gamma_n^2} F_\ell(k). \tag{9}$$

For the transparent potential generated by (5), the final Jost function is $\tilde{F}_{\ell+2N}(k) = 1$. It can be produced by the centrifugal barrier of the $\ell + 2N$ partial wave,

$$W_{\ell+2N}(r) = (\ell + 2N)(\ell + 2N + 1)/r^2. \tag{10}$$

Remembering that we are dealing with potentials in the ℓ-wave, the ℓ-wave effective potential that generates the Jost function $\tilde{F}_{\ell+2N}(k) = 1$ is

$$V_0(r) = \frac{(\ell + 2N)(\ell + 2N + 1)}{r^2} - \frac{\ell(\ell + 1)}{r^2} = \frac{2N(2\ell + 2N + 1)}{r^2}. \tag{11}$$

Therefore, we have to determine the family of potentials, depending on N normalization constants $C_1, C_2, \ldots, C_N$, such that when all normalization constants are put to infinity we recover the potential $V_0(r)$. One way to proceed is to apply successively N times, the supersymmetric transformations T_2 and T_4 of [7], to the centrifugal barrier $W_{\ell+2N}(r)$, equation (10). Indeed, the T_2 and T_4 transformations, with the appropriate parameter γ_p, introduce, successively, the multiplicative factors $(k - \mathrm{i}\gamma_p)/k$ and $(k + \mathrm{i}\gamma_p)/k$ in the Jost function while reducing the angular momentum from ℓ to $\ell - 1$ each time. However, an easier way to proceed is to resort to inverse scattering methods.

To determine the aforementioned phase-equivalent potentials, we employ the Marchenko inverse procedure. The normalization constants M_n involved in the Marchenko theory are known to be inversely proportional to the C_ns [1]. Thus, the infinite values of C_ns correspond to zeros of M_ns, which is much easier to handle numerically. In what follows we shall briefly recall aspects of the Marchenko method.

We assume that the underlying potential V_0 (dropping temporarily the previous definition (11)) satisfies the integrability conditions (1). The corresponding Jost solution $f_\ell^0(k, r)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} f_\ell^0(k, r) + \left(k^2 - V_0(r) - \frac{\ell(\ell + 1)}{r^2}\right) f_\ell^0(k, r) = 0 \tag{12}$$

and has the asymptotic behaviour

$$f_\ell^0(k, r) \underset{r\to\infty}{\longrightarrow} \mathrm{i}^\ell\, \mathrm{e}^{\mathrm{i}kr}. \tag{13}$$

Assume that V_0 supports n bound states, $E_n = -\hbar^2\, \gamma_n^2/2m$. The corresponding normalization constants involved are given by [1, 2]

$$\frac{1}{M_n^0} = \int_0^\infty f_\ell^0(\mathrm{i}\gamma_n, r)^2\, \mathrm{d}r. \tag{14}$$

We want to derive a new potential V having the same spectrum as V_0 but different in the normalization constants. Let M_n be the new normalization factors. We may then construct the kernel $A_0(r, t)$

$$A_0(r, t) = -\sum_{n=1}^{N} (M_n - M_n^0)\, f_\ell^0(\gamma_n, r)\, f_\ell^0(\gamma_n, t) \tag{15}$$

which behaves like $r^{\ell+1}$ and $t^{\ell+1}$ at $r \sim 0$ and $t \sim 0$, respectively, and decreases exponentially for large t. One defines the new kernel

$$A(r,t) = -\sum_{n=1}^{N}(M_n - M_n^0) f_\ell(\gamma_n, r) f_\ell^0(\gamma_n, t) \qquad t \geqslant r \tag{16}$$

which satisfies

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial t^2} + \frac{\ell(\ell+1)}{t^2}\right) A(r,t) = (V(r) - V_0(t))\, A(r,t). \tag{17}$$

The corresponding Jost solution is given by

$$f_\ell(k,r) = f_\ell^0(k,r) + \int_r^\infty A(r,s)\, f_\ell^0(k,s)\,\mathrm{d}s. \tag{18}$$

Since $A(r,t)$ decreases exponentially as $t \to \infty$, the function $f_\ell(k,r)$ is a solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} f_\ell(k,r) + \left(k^2 - V(r) - \frac{\ell(\ell+1)}{r^2}\right) f_\ell(k,r) = 0 \tag{19}$$

provided that

$$V(r) - V_0(r) = -2\frac{\mathrm{d}}{\mathrm{d}r} A(r,r). \tag{20}$$

Furthermore, one can show that

$$A(r,t) = A_0(r,t) + \int_r^\infty A(r,s)\, A_0(s,t)\,\mathrm{d}s \qquad t \geqslant r. \tag{21}$$

According to (21), $A(r,t)$ has the same behaviour as $A_0(r,t)$ at short distances r, i.e. $\sim r^{\ell+1}$. Furthermore, since $f_\ell^0(k,r) \simeq r^{-\ell}$ for $r \sim 0$, the integral in equation (18) converges for $r \to 0$, and has an r^2 behaviour. Therefore, for small r the behaviour of $f_\ell(k,r)$ is dominated by that of $f_\ell^0(k,r)$ (i.e. like $r^{-\ell}$) and the Jost function $F_\ell(k)$,

$$F_\ell(k) = \lim_{r\to 0} \frac{(-kr)^\ell}{(2\ell-1)!!}\, f_\ell(k,r) \tag{22}$$

is then equal to the original Jost function $F_\ell^0(k)$. Moreover, one can show that the normalization constants M_n are given by

$$\frac{1}{M_n} = \int_0^\infty f_\ell(\mathrm{i}\gamma_n, r)^2\,\mathrm{d}r \tag{23}$$

while the regular solution $\psi_\ell(k,r)$ of the Schrödinger equation is given by

$$\psi_\ell(k,r) = \frac{\mathrm{i}}{2k^{\ell+1}}\left[F_\ell(k)\, f_\ell(-k,r) - (-1)^\ell\, F_\ell(-k)\, f_\ell(k,r)\right] \tag{24}$$

and satisfies (7). The constants M_n are inversely proportional to the 'standard' normalization constants C_n obtained from (6) and (7), i.e. from $\psi_\ell(k,r)$. Their explicit relation is given by [1]

$$\frac{1}{C_n} = -\, M_n\, (-)^\ell\, \frac{1}{4\gamma_n^{2+2\ell}} \left(\frac{\mathrm{d}}{\mathrm{d}k} F_\ell(k)\right)^2 \Bigg|_{k=\mathrm{i}\gamma_n} \tag{25}$$

in terms of $F_\ell(k)$, the Jost function for the potential V.

For the starting potential V_0, equation (11), all the M_n^0 are identically zero. Adding to V_0 the ℓ-wave centrifugal term we obtain the potential $W_{\ell+2N}(r)$ since

$$V_0(r) + \ell(\ell+1)/r^2 = (\ell+2N)(\ell+2N+1)/r^2.$$

Therefore, the functions $f_\ell^0(k,r)$, denoted as in [1] by $w_{\ell+2N}(kr)$, are given by

$$w_{\ell+2N}(r) = \mathrm{i}(-)^\ell \sqrt{\frac{\pi kr}{2}}\, H^{(1)}_{\ell+2N+1/2}(r) \tag{26}$$

in terms of the Hankel function $H^{(1)}_\nu$ of the order of ν and of the first kind. At infinity we have the usual asymptotic behaviour $w_{\ell+2N}(kr) \simeq \mathrm{i}^{\ell+2N}\exp(\mathrm{i}kr)$.

The required potential associated with (5) is constructed using equations (15), (20) and (21). The solution of the Marchenko equation can easily be obtained numerically or by using the method described in appendix B.

3. Examples

Consider first the case with $N = 1$. We may set for convenience $\gamma \equiv \gamma_N$, $C \equiv C_N$, $M \equiv M_N = (-)^\ell \gamma^{4+2\ell}/C$, to obtain the simple expression (see appendix B)

$$V(r) = -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln(G) + \frac{2(2\ell+3)}{r^2} \tag{27}$$

with

$$G = 1 + \frac{(-)^\ell\gamma^{2+2\ell}}{2C}\bigg[r w'_{\ell+2}(\mathrm{i}\gamma r)^2 - w'_{\ell+2}(\mathrm{i}\gamma r)w_{\ell+2}(\mathrm{i}\gamma r) - r w^2_{\ell+2}(\mathrm{i}\gamma r)\left(\gamma^2 + \frac{(\ell+2)(\ell+2+1)}{r^2}\right)\bigg] \tag{28}$$

where the symbol prime denotes the derivative with respect to r. From equation (18), we obtain for $k \neq \mathrm{i}\gamma$ the Jost solution

$$f_\ell(k,r) = w_{\ell+2}(kr) - \frac{(-)^\ell\gamma^{4+2\ell}w_{\ell+2}(\mathrm{i}\gamma r)}{CG(\gamma^2+k^2)}\left[w'_{\ell+2}(kr)w_{\ell+2}(\mathrm{i}\gamma r) - w_{\ell+2}(kr)w'_{\ell+2}(\mathrm{i}\gamma r)\right] \tag{29}$$

and for $k = \mathrm{i}\gamma$

$$f_\ell(\mathrm{i}\gamma, r) = w_{\ell+2}(\mathrm{i}\gamma r)/G. \tag{30}$$

The Jost solution (29) and (30) behaves at infinity like $\mathrm{i}^{\ell+2}\exp(\mathrm{i}kr)$. Remembering that it is a Jost solution for the ℓ-wave, the Jost solution to be considered, correctly normalized at infinity according to (13), is, in fact, $-f_\ell(k,r)$. The corresponding (to $-f_\ell(k,r)$) regular wavefunction is given by (24). Taking into account the relation $f_\ell(k,r)^* = (-)^\ell f_\ell(-k,r)$, we obtain for $k \neq \mathrm{i}\gamma$

$$\psi_\ell(k,r) = \frac{(-)^\ell\,(k^2+\gamma^2)}{k^{\ell+3}}\,\mathrm{Im}\,(f_\ell(k,r)) \tag{31}$$

and thus

$$\psi_\ell(k,r) = -\frac{1}{k^{\ell+3}}\,\mathrm{Im}\bigg[\frac{\gamma^4}{CG}w_{\ell+2}(\mathrm{i}\gamma r)\big(w'_{\ell+2}(kr)w_{l+2}(\mathrm{i}\gamma r) - w_{\ell+2}(kr)\,w'_{\ell+2}(\mathrm{i}\gamma r)\big) + (-)^\ell\,(k^2+\gamma^2)w_{\ell+2}(kr)\bigg] \tag{32}$$

whereas for $k = \mathrm{i}\gamma$

$$\psi_\ell(\mathrm{i}\gamma, r) = \mathrm{i}^\ell \gamma^{\ell+2}\, \frac{w_{\ell+2}(\mathrm{i}\gamma r)}{CG}. \tag{33}$$

For the $\ell = 0$ partial wave we find that the potential is given by the explicit expression

$$V(r) = 2\frac{\mathcal{N}(r)}{\left(6 + 12\gamma r + 6\gamma^2 r^2 + \gamma^3 r^3 + 2Cr^3 \exp(2\gamma r)\right)^2} \tag{34}$$

with

$$\begin{aligned}\mathcal{N}(r) = {} & 72\gamma^2 - 72Cr\exp(2\gamma r) + 108\gamma^3 r - 144C\exp(2\gamma r)\gamma r^2 + 72\gamma^4 r^2 \\ & -192C\exp(2\gamma r)\gamma^2 r^3 + 24\gamma^5 r^3 + 12C^2\exp(4\gamma r) r^4 \\ & -132C\exp(2\gamma r)\gamma^3 r^4 + 3\gamma^6 r^4 - 48C\exp(2\gamma r)\gamma^4 r^5 - 8C\exp(2\gamma r)\gamma^5 r^6.\end{aligned} \tag{35}$$

We note that the SUSY-2 partner of V, obtained from equations (34) and (35) at the limit C infinite, is $6/r^2$, i.e. it has the behaviour of the $\ell = 2$ centrifugal barrier. For $\ell \neq 0$ its behaviour is $2(2\ell + 3)/r^2$. The Jost solution is then given by

$$\begin{aligned} f(k, r) = {} & \frac{\exp(\mathrm{i}kr)}{k^2\left(2Cr^3\exp(2\gamma r) + D(r)\right)}\Big[2C\exp(2\gamma r) r(k^2 r^2 + 3\mathrm{i}kr - 3) \\ & + \frac{k - \mathrm{i}\gamma}{k + \mathrm{i}\gamma}(D(r)k^2 + 3\mathrm{i}\gamma(2 + \gamma r)(k(2 + \gamma r) + \mathrm{i}\gamma))\Big] \end{aligned} \tag{36}$$

where

$$D(r) = 6 + 12\gamma r + 6\gamma^2 r^2 + \gamma^3 r^3. \tag{37}$$

It is noted that for $r = 0$ we obtain the required result $f(k, 0) = F(k) = (k^2 + \gamma^2)/k^2$.

Finally, the regular wavefunction is given by

$$\begin{aligned}\psi(k, r) = {} & \frac{k^2 + \gamma^2}{k^5(2Cr^3\exp(2\gamma r) + D(r))}(2Cr\exp(2\gamma r)(3kr\cos(kr) + (k^2 r^2 - 3)\sin(kr)) \\ & + (3\gamma^2(2 + \gamma r) - D(r)k^2)\sin(kr + u) - 3\gamma k(2 + \gamma r)^2\cos(kr + u))\end{aligned} \tag{38}$$

where

$$\exp(\mathrm{i}u) = \frac{(\gamma + \mathrm{i}k)^2}{\gamma^2 + k^2}. \tag{39}$$

We note that $\psi(k, r)$ satisfies the proper condition at the origin

$$\frac{\mathrm{d}}{\mathrm{d}r}\psi(k, r)|_{r=0} = 1. \tag{40}$$

4. Generalization

Up to now we have dealt with phase-equivalent potentials for which the S_ℓ-matrix is unity. In this section we shall investigate phase-equivalent potentials, starting from a non-zero potential, and show that the previous procedure can be extended, in a straightforward way, to cases where $S_\ell \neq 1$.

In line with the previous discussion we have to construct the potential, and the corresponding Jost solution which satisfies the property $\tilde{F}_{\ell+2N}(k) = F_\ell(k)$. When $F_\ell(k) = 1$ the total potential (potential + ℓth centrifugal barrier) and the Jost solution were the centrifugal barrier for $\ell + 2N$ and $w_{\ell+2N}(kr)$, respectively. In the general case, however, we need an explicit construction of both the potential and the Jost solution. For this we shall use the procedure described in [1] by increasing ℓ while leaving the Jost function unchanged. We first assume that the Jost function $F_\ell(k)$ has no zeros on the positive imaginary axis, i.e. the corresponding potential $V(r)$ does not support any bound states. Let $f(k,r)$ and $\psi(k,r)$ be the corresponding Jost and regular solutions, respectively, normalized according to (13) and (7) for the energy $E = \hbar^2k^2/2m$. Since $V(r)$ has no bound state, the regular solution at zero energy, labelled $\psi_0(r)$, never vanishes except for $r = 0$.

Following [1] we introduce

$$f^{(1)}(k,r) = \frac{f'(k,r)\psi_0(r) - f(k,r)\psi_0'(r)}{k\,\psi_0(r)} \qquad k \neq 0 \tag{41}$$

where the symbol prime denotes the derivative with respect to r, and

$$\psi^{(1)}(k,r) = \frac{\psi(k,r)\psi_0'(r) - \psi'(k,r)\psi_0(r)}{k^2\,\psi_0(r)} \qquad k \neq 0 \tag{42}$$

or, equivalently,

$$\psi^{(1)}(k,r) = \frac{\int_0^r \psi(k,x)\,\psi_0(x)\,\mathrm{d}x}{\psi_0(r)}. \tag{43}$$

From (43) we have

$$\psi_0^{(1)}(r) = \frac{\int_0^r \psi_0^2(x)\,\mathrm{d}x}{\psi_0(r)}. \tag{44}$$

The functions $f^{(1)}(k,r)$ and $\psi^{(1)}(k,r)$ can be obtained by solving the Schrödinger equations for the $(\ell+1)$th partial wave, with the potential

$$V^1(r) = V(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\psi_0(r)}{r^{\ell+1}}\right). \tag{45}$$

Note that $f^{(1)}$ and $\psi^{(1)}$ are correctly normalized according to (13) and (7). Indeed, in the absence of bound states at zero energy, the function $\psi_0(r)$ behaves like $r^{\ell+1}$ at the origin and infinity. Consequently,

$$f^{(1)}(k,r) \simeq \frac{f'(k,r)}{k} \simeq \mathrm{i}^{\ell+1}\,\mathrm{e}^{\mathrm{i}kr}$$

and $\psi^{(1)}$, as defined by (43), is properly normalized at the origin. As expected

$$f^{(1)}(k,r) \simeq \frac{(2\ell+1)!!}{(-kr)^{\ell+1}}\,F_\ell(k)$$

in the vicinity of zero.

As an example we consider the Jost function for the s-wave

$$F(k) = \frac{k + \mathrm{i}a}{k + \mathrm{i}b} \qquad b > 0. \tag{46}$$

The corresponding potential is

$$V(r) = -8b^2\beta \frac{\exp(-2br)}{\left(1 + \beta\exp(-2br)^2\right)} \tag{47}$$

and the Jost solution

$$f(k,r) = \mathrm{e}^{\mathrm{i}kr}\frac{k + \mathrm{i}b + \beta\exp(-2br)(k - \mathrm{i}b)}{(1 + \beta\exp(-2br))(k + \mathrm{i}b)} \tag{48}$$

where $\beta = (b-a)/(b+a)$.

Applying the transformation (41) twice, we obtain, after some algebra,

$$f^{(2)}(k,r) = -f(k,r) - \frac{\psi_0(r)}{k\,\psi_0^{(1)}(r)}\,f^{(1)}(k,r) \tag{49}$$

which satisfies (we keep the ℓ dependence for the sake of generality; in our example $\ell = 0$)

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f^{(2)}(k,r) + \left(k^2 - V^{(2)}(r) - \frac{(\ell+2)(\ell+3)}{r^2}\right)f^{(2)}(k,r) = 0. \tag{50}$$

The new potential is given by

$$V^{(2)}(r) = V(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\int_0^r \psi_0(x)^2\,\mathrm{d}x}{r^{2\ell+3}}\right). \tag{51}$$

For $\ell = 0$ and for the Jost function (46) we have

$$V^{(2)}(r) = -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(q^2(\cosh(br) + q\sinh(br)) + 3x(1-q^2)\frac{br\cosh(br) - \sinh(br)}{b^3r^3}\right) \tag{52}$$

and

$$\begin{aligned} f^{(2)}(k,r) = -\big\{&f(k,r)\left[(x^3-1)y^2 + 3\mathrm{i}qx(\mathrm{i}q + xy)(1+y^2)\right] \\ &-3\mathrm{i}q(y-\mathrm{i})(\mathrm{i}q+xy)^2\exp(\mathrm{i}kr)\big\} \\ &\times\big\{y^2(b^3q^3r^3 + 3qx(1-q^2)(br - \tanh(br))/(1+q\tanh(br))\big\}^{-1} \end{aligned} \tag{53}$$

with $f(k,r)$ given by (48) and

$$q = \frac{a}{b} \qquad y = \frac{k}{b} \qquad x = 1 + bqr. \tag{54}$$

For r close to zero, $x \simeq 1 + 3bqr$. Thus

$$f^{(2)}(k,r) \simeq \frac{3bqr(\mathrm{i}q + y)}{y^2(\mathrm{i}+y)b^3r^3q} = \frac{3}{(kr)^2}\,\frac{k + \mathrm{i}a}{k + \mathrm{i}b}.$$

In general, and for a non-zero starting potential, phase-equivalent potentials can be obtained by substituting the functions $f^{(\ell+2N)}$ for $w_{\ell+2N}$ and by replacing the potential $V_0(r)$, equation (11), by $V^{(\ell+2N)}(r) + V_0(r)$. For $N = 1$ we obtain the phase-equivalent potentials, labelled $V_{\text{final}}(r)$

$$V_{\text{final}}(r) = -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln(\tilde{G}) + V^{(2)}(r) + \frac{2(2\ell+3)}{r^2} \tag{55}$$

where

$$\tilde{G} = 1 + M\int_r^\infty f^{(2)}(\mathrm{i}\gamma, t)^2\,\mathrm{d}t. \tag{56}$$

For the potential (47) and for $\ell = 0$, $M = \gamma^4(b+\gamma)^2/(C(a+\gamma)^2)$ and thus one finally obtains

$$\begin{aligned} V_{\text{final}}(r) &= -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\mathcal{D} \\ \mathcal{D} &= 2C\frac{(a+\gamma)^2}{(b+\gamma)^2}\mathrm{e}^{2\gamma r}\left(q^2r^3(\mathrm{e}^{2br}+\beta) + 3(1-q)\left(f_-(r)\,\mathrm{e}^{2br} + f_+(r)\right)\right) \\ &\quad + D(r)q^2(\mathrm{e}^{2br} + \beta\,\eta^2) + 3\gamma^3(1-q)(f_-(w)\mathrm{e}^{2br} + \eta^2 f_+(w)) \end{aligned} \tag{57}$$

where $D(r)$ is given by (37) and

$$w = \frac{2+\gamma r}{\gamma} \qquad q = \frac{a}{b} \qquad \beta = \frac{b-a}{b+a} \qquad \eta = \frac{b-\gamma}{b+\gamma} \tag{58}$$

while $f_\pm(x)$ is given by

$$f_\pm(x) = \frac{\pm 1 + bx(1\pm q) + b^2q\,x^2}{b^3}. \tag{59}$$

When $q = a/b = 1$ ($\beta = 0$) we are left with

$$V_{\text{final}}(r) = -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(2Cr^3\mathrm{e}^{2\gamma r} + \gamma^3r^3 + 6\gamma^2r^2 + 12\gamma r + 6\right) \tag{60}$$

and we recover the transparent potential equation (34).

The Jost solution for $V_{\text{final}}(r)$ and for $\ell = 0$, once correctly normalized at infinity, reads

$$\begin{aligned} f_{\text{final}}(k,r) = -\Bigg[& f^{(2)}(k,r)\left(1 + M\int_r^\infty f(\mathrm{i}\gamma,t)^2\,\mathrm{d}t\right) - f^{(2)}(\mathrm{i}\gamma,r)M\int_r^\infty f(k,t)f(\mathrm{i}\gamma,t)\,\mathrm{d}t \\ & -M\frac{\psi_0(r)}{\psi_0^{(1)}(r)}\frac{f^{(1)}(\mathrm{i}\gamma,r)}{\mathrm{i}\gamma}\left(f(k,r)\frac{f^{(1)}(\mathrm{i}\gamma,r)}{\mathrm{i}\gamma} - f(\mathrm{i}\gamma,r)\frac{f^{(1)}(k,r)}{k}\right)\Bigg]\Big/\tilde{G} \end{aligned} \tag{61}$$

with $\tilde{G}$ being given by (56).

For the Jost function (46) and $\ell = 0$ we have

$$f_{\text{final}}(k,r) = \mathrm{e}^{\mathrm{i}kr}\frac{\Lambda}{k^2\,\mathcal{D}(r)} \tag{62}$$

with

$$\begin{aligned} \Lambda = 2C\frac{(a+\gamma)^2}{(b+\gamma)^2}\mathrm{e}^{2\gamma r}\Bigg[& q^2k^2r^3\left(\mathrm{e}^{2br} + \beta\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right) + \frac{3}{b^3}(1-q)(k-\mathrm{i}b)(k+\mathrm{i}a)(1-\mathrm{e}^{2br}) \\ & + 3\left(F_-(r)\mathrm{e}^{2br} + F_+(r)\beta\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)\Bigg] \end{aligned}$$

$$+\frac{k-\mathrm{i}\gamma}{k+\mathrm{i}\gamma}\left[D(r)k^2q^2\left(\mathrm{e}^{2br}+\beta\,\eta^2\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)\right.$$

$$+\frac{3\gamma^3}{b^3}(1-q)(k-\mathrm{i}b)(k+\mathrm{i}a)(\eta^2-\mathrm{e}^{2br})$$

$$\left.+3\gamma^3\left(F_-(w)\mathrm{e}^{2br}+F_+(w)\beta\eta^2\frac{k-\mathrm{i}b}{k+\mathrm{i}b}\right)\right] \tag{63}$$

where $\mathcal{D}$ is given by (57) and

$$F_\pm(x)=\frac{x(k\pm kq+\mathrm{i}bq)(k\pm kq+\mathrm{i}bq+bkqx)}{b^2}.$$

The function $f_{\text{final}}(k,r)$ is normalized at infinity according to (13). For $r=0$ ($w=2/\gamma$, $D(r=0)=6$) we obtain $f_{\text{final}}(k,0)=(k^2+\gamma^2)(k+\mathrm{i}a)/(k^2(k+\mathrm{i}b))$. When $a=b$, i.e. for $q=1$, $\beta=0$ we recover the Jost function equation (36).

The previous construction assumes that the starting potential has no bound state. When the potential indeed supports a bound state, i.e. when the Jost function is given by

$$F(k)=\frac{k-\mathrm{i}a}{k+\mathrm{i}b}\qquad b>0 \tag{64}$$

then the introduction of a unitary factor leads to the Jost function (46) from which we construct $f_{\text{final}}(k,r)$ and the corresponding regular solution $\psi_{\text{final}}(k,r)$ via equation (24). Using, then, the Gel'fand–Levitan formalism we obtain the potential

$$\tilde{V}(r)=V_{\text{final}}(r)-2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(1+C_a\int_0^r\psi_{\text{final}}(\mathrm{i}a,t)^2\,\mathrm{d}t\right) \tag{65}$$

in terms of the normalization constant C_a of the bound state $-a^2\hbar^2/2m$.

When the ghost component is allowed to have spectral singularities at the points $k=\pm b$, for example, if we consider

$$F_\ell^G(k)=\frac{k^2+\gamma^2}{k^2-b^2}\qquad k\neq b\quad b>0 \tag{66}$$

we can apply the previous procedure. For instance, the transparent s-wave potentials, generated by (66), are obtained as follows. First, we construct the function

$$f^{(2)}(k,r)=-\frac{\exp(\mathrm{i}kr)}{k^2-b^2}\frac{[(k-b)^2X^2-(k+b)^2+4bX(k+\mathrm{i}r(b^2-k^2))]}{(X^2-4\mathrm{i}brX-1)}\qquad k\neq b \tag{67}$$

where $X=\exp(2\mathrm{i}br)$. For r close to zero, it behaves like $3/(r^2(k^2-b^2))$. Then applying (55) and (61), we find the potential

$$V(r)=-2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln(D^b)$$

$$D^b=(\sin(2br)-2br)(b^4-6b^2\gamma^2+\gamma^4+2\gamma C\mathrm{e}^{2\gamma r})$$

$$+(\cos(2br)-1)(4b\gamma(\gamma^2-b^2))-8b^3\gamma(1+2\gamma r)$$

which behaves asymptotically as $-8b^2\sin(2br)/(2br)$.

The associated Jost solution reads

$$f(k,r) = \frac{\exp(\mathrm{i}kr)}{k^2-b^2}\,\frac{N^b}{\tilde{D}^b}$$

$$N^b = 2\gamma C\mathrm{e}^{2\gamma r}\,[(k-b)^2\,X^2 - (k+b)^2 + 4bX\,(k+\mathrm{i}r(b^2-k^2))]$$

$$-\frac{k-\mathrm{i}\gamma}{k+\mathrm{i}\gamma}[(b+k)^2(b+\mathrm{i}\gamma)^4 - X^2\,(b-k)^2(b-\mathrm{i}\gamma)^4$$

$$+4\mathrm{i}bX(b^2+\gamma^2)(2\gamma(k^2-b^2)+(b^2+\gamma^2)(\mathrm{i}k-r(b^2-k^2))]$$

$$\tilde{D}^b = 2\gamma C\mathrm{e}^{2\gamma r}(X^2-4\mathrm{i}brX-1)$$

$$-[(b+\mathrm{i}\gamma)^4 - X^2(b-\mathrm{i}\gamma)^4 + 4\mathrm{i}bX(b^2+\gamma^2)(2\gamma+r(b^2+\gamma^2))].$$

5. Conclusion

The supersymmetric (Darboux) factorizations have elucidated several aspects of effective interactions, especially those related to deep potentials and of their shallower SUSY partners. The SUSY transformations together with the inverse scattering problem at fixed angular momentum, inherently related to them [7], provide us with various classes of phase or phase and spectrum-equivalent potentials. In the inverse scattering method of Marchenko, for example, the basic input is the S-matrix whose poles can be interpreted either as bound states, i.e. zeros of the Jost function, or as stemming from the poles of $F_\ell(-k)$. When all poles of the S-matrix are considered as being poles of $F(-k)$ the resulting potential is then the shallowest, phase-equivalent, SUSY partner. This latter potential is unique and can be obtained by setting the asymptotic normalizations, present in the Marchenko fundamental equation, equal to zero. Alternatively, if we consider that some of the poles correspond to genuine bound states, we may apply successive sets of SUSY transformations to remove the non-physical states [8].

We note that once the spurious (non-physical) states are removed the potential behaves as $2N(2N+2\ell+1)/r^2$ at short distances, N being the number of spurious bound states eliminated. This repulsion at the origin is due to the definition of the SUSY potentials, which involves the logarithmic derivative of the regular wavefunction in their structure.

In this paper we have investigated yet another aspect of the inverse scattering problem or, equivalently, of supersymmetric transformations. This is related to the introduction of ghost components in the Jost function, i.e. components which, although present in the Jost function, cancel out in the S-matrix and thus are not apparent in any way in the experimental data. The ghost components do provide zeros in the Jost function and one would normally expect that these would result in bound states. The fact, however, that these states are not seen by experiment, suggests that the associated family of phase-equivalent potentials supports non-physical bound states. When each spurious or non-physical bound state is removed, by setting the corresponding normalization constants in the Marchenko equations equal to zero, the correct interaction appears as the unique shallow SUSY partner.

In this paper we have considered two distinct cases related to the ghost components, which are assumed to have the form of a rational function. The first case arises when the S_ℓ-matrix is equal to unity for all momenta. The resulting new transparent potentials are of the Bargmann class. In the second case, the starting S_ℓ-matrix is different from unity. In both cases a new class of phase-equivalent potentials can be constructed, for each angular momentum, having the same behaviour at the origin as the original potential but characterized by a long-ranged tail, i.e. an asymptotic behaviour in $2N(2N + 2\ell + 1)/r^2$, according to the number N of ghost components introduced.

A final remark should be made concerning the role of ℓ and the actual angular momentum of the system. As a result of our transformation there is a shift of the partial wave from ℓ to $\ell+2$ for example, and many of our expressions indeed suggest a change of angular momentum. This is deceptive as we are dealing with the radial Schrödinger equation and the construction of ℓ-dependent phase-equivalent potentials which must be referred to the same partial wave and have the same phase shifts within this specific ℓ. Keeping this in mind, we should be careful not to interpret the appearance of a repulsive singularity $2N(2N + 2\ell + 1)/r^2$ in SUSY partners as a change of the angular momentum of the system.

Acknowledgments

Financial support from the National Research Foundation of South Africa is greatly appreciated. Two of us (SYL and SAS) acknowledge the hospitality of IPN-Orsay (IN2P3-CNRS) where part of this work was carried out.

Appendix A. Mathematical details

To prove the theorem of section 2, we use the Gel'fand–Levitan equations to change the normalization constant C_p into some value C. The regular wavefunction $\psi_\ell(k, r)$ fulfilling the Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_\ell(k, r) + \left(k^2 - V(r) - \frac{\ell(\ell + 1)}{r^2}\right)\psi_\ell(k, r) = 0 \tag{A1}$$

with

$$(2\ell + 1)!! \lim_{r\to 0} r^{-\ell-1}\psi_\ell(k, r) = 1 \tag{A2}$$

transforms into (see [1], equation (IV.2.2))

$$\tilde{\psi}_\ell(k, r) = \psi_\ell(k, r) - \frac{(C - C_p)\psi_\ell(\mathrm{i}\gamma_p, r)}{1 + (C - C_p)\int_0^r \psi_\ell^2(\mathrm{i}\gamma_p, x)\,\mathrm{d}x}\int_0^r \psi_\ell(k, x)\psi_\ell(\mathrm{i}\gamma_p, x)\,\mathrm{d}x. \tag{A3}$$

where

$$\int_0^\infty \psi(\mathrm{i}\gamma_p, r)^2\,\mathrm{d}r = \frac{1}{C_p}. \tag{A4}$$

When C is finite, $\tilde{\psi}_\ell$ behaves like $r^{\ell+1}$ at the origin. This is no longer the case for C infinite.

Indeed, let us consider the Volterra equation

$$\psi_\ell(k, r) = \frac{r^{\ell+1}}{(2\ell + 1)!!} + \frac{1}{2\ell + 1}\int_0^r \left(\frac{r^{\ell+1}}{r'^{\ell}} - \frac{r'^{\ell+1}}{r^{\ell}}\right)(V(r') - k^2)\psi_\ell(k, r')\,\mathrm{d}r'. \tag{A5}$$

When $V(r)$ is such that at the origin $V(r) = V(0)+r\mathcal{O}(r)$, replacing $\psi_\ell(k, r')$ by $r'^{\ell+1}/(2\ell+1)!!$ in the integral involved in (A5), leads, when $r \sim 0$, to

$$\psi_\ell(k, r) = \frac{r^{\ell+1}}{(2\ell + 1)!!}\left(1 + \frac{r^2}{2(2\ell + 3)}(V(0) - k^2)\right). \tag{A6}$$

Introducing (A6) in (A3) we obtain

$$\tilde{\psi}_\ell(k,r) = \frac{r^{\ell+1}}{(2\ell+1)!!} + \mathcal{O}(r^{\ell+3}). \tag{A7}$$

When $C \to \infty$ we have

$$\tilde{\psi}_\ell(k,r) = -\frac{r^{\ell+3}}{(2\ell+5)!!}(k^2+\gamma_p^2) + \mathcal{O}(r^{\ell+5}) \tag{A8}$$

and the regular wavefunction has behaviour corresponding to angular momentum ℓ+2, provided the corresponding potential is finite at the origin.

Let us elaborate on this change of angular momentum. First, we introduce

$$\tilde{\psi}^R_{\ell+2}(k,r) = \begin{cases} -\tilde{\psi}_\ell(k,r)/(k^2+\gamma_p^2) & k \neq \mathrm{i}\gamma_p \\ -\lim\limits_{k\to\mathrm{i}\gamma_p} \tilde{\psi}_\ell(k,r)/(k^2+\gamma_p^2) & k = \mathrm{i}\gamma_p. \end{cases} \tag{A9}$$

This renormalized function satisfies

$$(2(\ell+2)+1)!! \lim_{r\to 0} r^{-\ell-2-1}\tilde{\psi}^R_{\ell+2}(k,r) = 1 \tag{A10}$$

and has a regular behaviour in the vicinity of zero for $\ell+2$ (see A2). It corresponds to the potential for the $(\ell+2)$ partial wave

$$\tilde{V}^R(r) = V(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left[\frac{\int_0^r \psi_\ell^2(\mathrm{i}\gamma_p, x)\,\mathrm{d}x}{r^{2\ell+3}}\right]. \tag{A11}$$

It is easy to see that $\tilde{V}^R(r)$ behaves as $1/r^2$ asymptotically.

Let us consider the asymptotic expressions for $\tilde{\psi}_\ell$ and $\tilde{\psi}^R_{\ell+2}$. From (A3) and for $r\to\infty$ the asymptotic behaviour of $\tilde{\psi}_\ell(k,r)$ is the same as that for $\psi_\ell(k,r)$, whether C finite or infinite

$$\tilde{\psi}_\ell(k,r) \to \frac{|F_\ell(k)|}{k^{\ell+1}}\sin(kr - \ell\pi/2 + \delta_\ell(k)) \tag{A12}$$

in terms of the Jost function of the potential $V(r)$ for the ℓ-partial wave. By definition, the asymptotic form of the regular wavefunction $\tilde{\psi}^R_{\ell+2}$ is given by

$$\tilde{\psi}^R_{\ell+2}(k,r) \to \frac{|F^R_{\ell+2}(k)|}{k^{\ell+3}}\sin(kr - (\ell+2)\pi/2 + \delta^R_{\ell+2}(k)) \tag{A13}$$

where the Jost function $F^R_{\ell+2}(k)$ corresponds to the potential $\tilde{V}^R$. Combining equations (A9), (A12) and (A13) we have

$$|F^R_{\ell+2}(k)| = |F_\ell(k)|\frac{k^2}{k^2+\gamma_p^2} \tag{A14}$$

and

$$\delta^R_{\ell+2}(k) = \delta_\ell(k). \tag{A15}$$

This implies that

$$F^R_{\ell+2}(k) = |F_\ell(k)|\frac{k^2}{k^2+\gamma_p^2}\mathrm{e}^{-\mathrm{i}\delta_\ell(k)} = F_\ell(k)\frac{k^2}{k^2+\gamma_p^2}. \tag{A16}$$

Appendix B. Solution of the Marchenko equation

The Marchenko equation can be solved using the determinants of the $N \times N$ matrix R defined by

$$R_{m,j} = \delta_{m,j} + M_m \int_r^\infty w_{\ell+2N}(\mathrm{i}\gamma_m x) w_{\ell+2N}(\mathrm{i}\gamma_j x)\, \mathrm{d}x \qquad m, j \leqslant N. \tag{B1}$$

The Jost solution is then given by $f_\ell(k, r) = \det(\tilde{R})/\det(R)$ where $\tilde{R}$ is the $(N+1) \times (N+1)$ matrix

$$\begin{aligned}
\tilde{R}_{m,j} &= R_{m,j} && m, j \leqslant N \\
\tilde{R}_{m,N+1} &= M_m w_{\ell+2N}(\mathrm{i}\gamma_m r) && m \leqslant N \\
\tilde{R}_{N+1,j} &= \int_r^\infty w_{\ell+2N}(\mathrm{i}\gamma_j x) w_{\ell+2N}(kx)\, \mathrm{d}x && j \leqslant N \\
\tilde{R}_{N+1,N+1} &= w_{\ell+2N}(kr).
\end{aligned} \tag{B2}$$

For large values of r, $f_\ell(k, r)$ has the behaviour $w_{\ell+2N}(kr) \simeq \mathrm{i}^{\ell+2N} \exp(\mathrm{i}kr)$. Remembering that $f_\ell(k, r)$ is a Jost solution for the ℓ-wave, the Jost solution, correctly defined at infinity is, in fact, $(-)^N f_\ell(k, r)$. Therefore,

$$f_\ell(k, r) = (-)^N \frac{\det(\tilde{R})}{\det(R)}. \tag{B3}$$

The regular solution $\psi_\ell(k, r)$ is according to (24) obtained by making the substitution in the last line of $\tilde{R}$

$$w_{\ell+2N} \mapsto \frac{\mathrm{i}(-)^N}{2k^{\ell+1}} \prod_{n=1}^{N} \frac{k^2 + \gamma_n^2}{k^2} \big(w_{\ell+2N}(-kr) - (-)^l w_{\ell+2N}(kr)\big). \tag{B4}$$

The potential is given in terms of M_n related to the C_n by (25). Here

$$M_n = (-)^\ell \gamma_n^{2\ell+4} \left(\prod_{m \neq n} \frac{\gamma_n^2 - \gamma_m^2}{\gamma_n^2} \right)^{-2} \frac{1}{C_n}. \tag{B5}$$

The potential V is then simply

$$V(r) = -2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} [\ln(\det R)] + \frac{2N(2N + 2\ell + 1)}{r^2}. \tag{B6}$$

For the special case with $N = 1$ we have

$$f_\ell(\mathrm{i}\gamma_1, r) = -\frac{w_{\ell+2}(\mathrm{i}\gamma_1 r)}{1 + M_1 \int_r^\infty w_{\ell+2}(\mathrm{i}\gamma_1 s)^2\, \mathrm{d}s} \tag{B7}$$

and thus

$$\frac{1}{M_1} = \int_0^\infty f_\ell(\mathrm{i}\gamma_1, r)^2\, \mathrm{d}r.$$

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